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Solutions for transversely isotropic piezoelectric infinite body, semi-infinite body and bimaterial infinite body subjected to uniform ring loading and charge

Ding Haojiang[a,]*, Chi Yuwei[b], Guo Fenglin[a]

[a] *Department of Civil Engineering, Zhejiang University, Hangzhou 310027, P.R. China*
[b] *Department of Geotechnical Engineering, Tongji University, Shanghai 200092, P.R. China*

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Abstract

Based on the fundamental solutions for transversely isotropic piezoelectric materials, the fundamental solutions of axisymmetric problems are derived by integration and explicit expressions for three possible cases of different characteristic roots and multiple roots are all presented. In the case of $s_1 \neq s_2 \neq s_3 \neq s_1$, based on the Green's functions for semi-infinite piezoelectric body and bimaterial infinite piezoelectric body, the Green's functions for axisymmetric problems of semi-infinite body and bimaterial infinite body are obtained. Taking PZT-4 as an example, numerical computations are conducted by use of the fundamental solutions to axisymmetric problems. Comparison of the calculated results with those of FEM shows good agreement between them.

1. Introduction

The problem of axisymmetric stress analysis of body of revolution is of great significance in engineering. Kermanidis (1975) and Cruse et al. (1977) studied the Boundary Integral Equation approach for axisymmetric problem of bodies of revolution. Rizzo and Shippy (1979) and Mayr et al. (1980) successfully extended this method to axisymmetric bodies with arbitrary boundary conditions. Brebbia et al. (1984) gave a detailed account of problems relating to application of BEM to axisymmetric bodies. For transversely isotropic materials, Hanson and Yang Wang (1997) recently gave solutions for ring loading in an infnite body and a semi-infinite body, including axial, radial and tangential loads. As for transversely isotropic piezoelectric materials, Ding et al. (1996) and Dunn and Wienecke (1996) gave three-dimensional fundamental solutions using different methods. Ding et al. (1997) obtained Green's function solutions for infinite, semi-infinite and bimaterial infinite bodies in all three cases of characteristic roots s_i.

* Corresponding author.

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In this paper, the fundamental solutions of axisymmetric problems are derived by integration methods based on fundamental solutions for transversely isotropic piezoelectric media, and explicit expressions are presented for three cases of different characteristic roots and multiple roots. For the case of $s_1 \neq s_2 \neq s_3 \neq s_1$, Green's functions for axisymmetric problems of semi-infinite body and bimaterial infinite body are obtained by integration method based on the Green's functions for semi-infinite piezoelectric body and bimaterial infinite piezoelectric body. Taking PZT-4 as an example, numerical computations are conducted by use of the fundamental solutions. Comparison of the calculated results with those of FEM shows good agreement between them. The notions in Ding et al. (1997) are widely adopted in the paper.

2. Solutions for uniform ring loading in an infinite body

Assume that xy plane is the isotropic plane. The coordinate system is shown in Fig. 1. Meanwhile, a cylindrical coordinate system (r, θ, z) is taken to coincide with the Cartesian coordinate system in z axis and the origin. Uniform ring loading and line charge density are applied at the ring $r = r_0$ on the plane $z = 0$. Elastic and electric fields caused by the loading and charge at an arbitrary point are intended to obtain. Without loss of generality, we assume that coordinates of the field point B are $B(r, 0, z)$, and coordinates of an arbitrary source point A in cylindrical and Cartesian coordinates are $(r_0, \theta, 0)$ and $(r_0 \cos\theta, r_0 \sin\theta, 0)$, respectively. In the following, attention should be paid to that in Ding et al. (1997) the source point is $(0, 0, h)$, thus the vector from source point to field point (x, y, z) is $(x, y, z-h)$. Here, the vector **AB** is $(r-r_0\cos\theta, -r_0\sin\theta, z)$ as shown in Fig. 1.

2.1. $s_1 \neq s_2 \neq s_3 \neq s_1$

(1) Solution for uniform ring loading in z direction with line density of P_l and uniform charge with line density of Q_l.

Assume that uniform ring loading in z direction with line density of P_l and uniform charge with line density of Q_l are applied at a ring passing through point A. Consider an infinitesimal arc element $r_0\,d\theta$ at point A, then the point force in z direction and point charge acting on the arc element are $P_l r_0\,d\theta$ and $Q_l r_0\,d\theta$, respectively. From eqns (12)–(14) of Ding et al. (1997), displacements at point B can be obtained as follows:

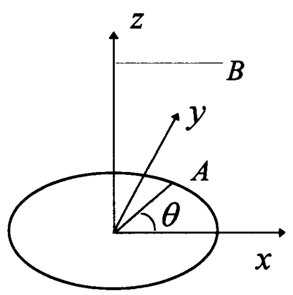

Fig. 1. Coordinate systems for axisymmetric problems.

$$\mathrm{d}u_r = \mathrm{sign}(z)\sum_{i=1}^{3}\frac{\bar{A}_i(rr_0 - r_0^2\cos\theta)}{\tilde{R}_i\tilde{T}_i}\mathrm{d}\theta \tag{1}$$

$$\mathrm{d}u_\theta = -\mathrm{sign}(z)\sum_{i=1}^{3}\frac{\bar{A}_i r_0^2\sin\theta}{\tilde{R}_i\tilde{T}_i}\mathrm{d}\theta \tag{2}$$

$$\mathrm{d}w_m = \sum_{i=1}^{3}\frac{\alpha_{im}\bar{A}_i r_0}{\tilde{R}_i}\mathrm{d}\theta, \quad (m = 1,2) \tag{3}$$

where w_1 is the displacement component in z directions, w, w_2 is electric potential ϕ and $\bar{A}_i = P_l A_i^P + Q_l A_i^Q$ with A_i^P and A_i^Q being constants. Constants A_i^P and A_i^Q can be determined by eqn (26) of Ding et al. (1997), in which $A_i = PA_i^P + QA_i^Q$, and

$$\tilde{R}_i = \sqrt{(r - r_0\cos\theta)^2 + r_0^2\sin^2\theta + s_i^2 z^2}, \quad \tilde{T}_i = \tilde{R}_i + s_i|z|, \quad (i = 0,1,2,3) \tag{4}$$

Using eqns (C5)–(C8) in Appendix C, eqn (1) can be rewritten to the following form:

$$\mathrm{d}u_r = \sum_{i=1}^{3}\frac{\bar{A}_i r_0}{r}\left[\frac{1}{2}\mathrm{sign}(z) - \mathrm{sign}(z)\frac{r^2 - r_0^2}{c^2 - b^2\cos\theta} - \frac{z_i}{2\tilde{R}_i} - \frac{(r^2 - r_0^2)z_i}{(c^2 - b^2\cos\theta)\tilde{R}_i}\right]\mathrm{d}\theta$$

Then, integrating the above expression with respect to θ in the interval of $0 \sim 2\pi$ and resorting to definite integral expressions eqns (C1)–(C4) of Appendix C, the representation of u_r can be readily obtained by integration. Integrals in eqns (2) and (3) are easy to integrate. Thus, we obtain displacements and electric potential as follows:

$$u_r = -\sum_{i=1}^{3}\frac{2\bar{A}_i r_0 z_i}{l_i r}\left[K(k_i) + \frac{r - r_0}{r + r_0}\Pi(d,k_i)\right], \quad u_\theta = 0$$

$$w_m = \sum_{i=1}^{3}\frac{4\alpha_{im}\bar{A}_i r_0}{l_i}K(k_i) \tag{5}$$

In the process of integration leading to eqn (5), the result $\Sigma_{i=1}^{3}\bar{A}_i = 0$ has been used since it is obvious that $\Sigma_{i=1}^{3}A_i^P = 0$ and $\Sigma_{i=1}^{3}A_i^Q = 0$ hold by eqn (26) of Ding et al. (1997).

It is not difficult to derive the expressions of strains, electric field strengths, stresses and electric displacements from eqn (5).

$$s_r = \sum_{i=1}^{3}\frac{2\bar{A}_i r_0 z_i}{l_i}\left[\frac{2}{g_i}E(k_i) + \frac{1}{r^2}K(k_i) + \frac{r - r_0}{(r + r_0)r^2}\Pi(d,k_i)\right]$$

$$s_\theta = -\sum_{i=1}^{3}\frac{2\bar{A}_i r_0 z_i}{l_i}\left[\frac{1}{r^2}K(k_i) + \frac{r - r_0}{(r + r_0)r^2}\Pi(d,k_i)\right]$$

$$s_{rz} = \sum_{i=1}^{3}\frac{2(\alpha_{i1} + s_i)\bar{A}_i r_0}{l_i r}\left[\frac{f_i}{g_i}E(k_i) - K(k_i)\right]$$

$$s_z = -\sum_{i=1}^{3} \frac{4\alpha_{i1}\bar{A}_i s_i r_0 z_i}{l_i g_i} E(k_i)$$

$$E_r = -\sum_{i=1}^{3} \frac{2\alpha_{i2}\bar{A}_i r_0}{l_i r} \left[\frac{f_i}{g_i} E(k_i) - K(k_i)\right]$$

$$E_z = \sum_{i=1}^{3} \frac{4\alpha_{i2}\bar{A}_i s_i r_0 z_i}{l_i g_i} E(k_i)$$

$$\sigma_r = (c_{11}-c_{12}) \sum_{i=1}^{3} \frac{2\bar{A}_i r_0 z_i}{l_i} \left[\frac{2}{g_i} E(k_i) + \frac{1}{r^2} K(k_i) + \frac{r-r_0}{(r+r_0)r^2} \Pi(d,k_i)\right] - \sum_{i=1}^{3} \frac{4\xi_i \bar{A}_i r_0 z_i}{l_i g_i} E(k_i)$$

$$\sigma_\theta = -(c_{11}-c_{12}) \sum_{i=1}^{3} \frac{2\bar{A}_i r_0 z_i}{l_i r^2} \left[K(k_i) + \frac{r-r_0}{r+r_0} \Pi(d,k_i)\right] - \sum_{i=1}^{3} \frac{4\xi_i \bar{A}_i r_0 z_i}{l_i g_i} E(k_i)$$

$$\sigma_m = -\sum_{i=1}^{3} \frac{4\vartheta_{im}\bar{A}_i r_0 z_i}{l_i g_i} E(k_i)$$

$$\tau_{rm} = \sum_{i=1}^{3} \frac{2\omega_{im}\bar{A}_i r_0}{l_i r} \left[\frac{f_i}{g_i} E(k_i) - K(k_i)\right] \tag{6}$$

where $m = 1, 2$, σ_1, σ_2, τ_{r1} and τ_{r2} stand for σ_z, D_z, τ_{rz} and D_r, respectively. For ξ_i, ω_{im} and ϑ_{im} $(i = 1, 2, 3)$, see eqn (7) of Ding et al. (1997).

(2) Solution for uniform ring loading in r direction with line density of T_l.

Consider an infinitesimal arc element $r_0\,\mathrm{d}\theta$ at point A. Then, the arc element is subjected to force in x direction $T_l r_0 \cos\theta\,\mathrm{d}\theta$ and force in y direction $T_l r_0 \sin\theta\,\mathrm{d}\theta$. Obviously, the displacement functions of point B can be obtained by superimposing the displacement functions of point force solution for force in x direction on those for force in y direction.

In Ding et al. (1997), eqn (35) gives the displacement functions for a point force T acting along x direction.

$$\psi_0 = \frac{D_0 y}{R_0 + s_0|z|}, \quad \psi_i = \frac{D_i x}{R_i + s_0|z|}, \quad (i = 1, 2, 3) \tag{7}$$

Similarly, the displacement functions for a point force T acting in y direction are:

$$\psi_0 = -\frac{D_0 x}{R_0 + s_0|z|}, \quad \psi_i = \frac{D_i y}{R_i + s_i|z|}, \quad (i = 1, 2, 3) \tag{8}$$

where D_i have been given by eqn (50) of Ding et al. (1997). Denote $T_l D_i/T$ $(i = 0, 1, 2, 3)$ as $\bar{D}_i$. By use of superposition principle and eqns (7) and (8), we have

$$\mathrm{d}\psi_0 = -\frac{\bar{D}_0 r r_0 \sin\theta}{\tilde{T}_0}\mathrm{d}\theta, \quad \mathrm{d}\psi_i = \frac{\bar{D}_i(r r_0 \cos\theta - r_0^2)}{\tilde{T}_i}\mathrm{d}\theta \tag{9}$$

Integrating the equations above with respect to θ in the interval $0 \sim 2\pi$ leads to the expressions of displacement functions:

$$\psi_0 = 0$$

$$\psi_i = 2\bar{D}_i\left\{-\pi[1-\mathrm{sign}(r-r_0)]s_i|z| - l_i E(k_i) + \frac{r^2-r_0^2}{l_i}K(k_i) + \frac{(r-r_0)z_i^2}{(r+r_0)l_i}\Pi(d,k_i)\right\} \tag{10}$$

Substituting eqn (10) into eqn (D1) and using eqns (B4)–(B7) give the expressions of displacements and electric potential.

$$u_r = -\sum_{i=1}^{3}\frac{2\bar{D}_i}{l_i r}[l_i^2 E(k_i) - n_i K(k_i)], \quad u_\theta = 0$$

$$w_m = \sum_{i=1}^{3}\frac{2\alpha_{im}\bar{D}_i z_i}{l_i}\left[K(k_i) - \frac{r-r_0}{r+r_0}\Pi(d,k_i)\right] \tag{11}$$

Furthermore, the expressions of strains, electric field strengths, tresses and electric displacements can be obtained:

$$s_r = \sum_{i=1}^{3}\frac{2\bar{D}_i}{l_i r^2}\left[\frac{p_i}{g_i}E(k_i) - (r_0^2+z_i^2)K(k_i)\right]$$

$$s_\theta = -\sum_{i=1}^{3}\frac{2\bar{D}_i}{r^2}\left[l_i E(k_i) - \frac{n_i}{l_i}K(k_i)\right]$$

$$s_{rz} = -\sum_{i=1}^{3}\frac{2(\alpha_{i1}+s_i)\bar{D}_i z_i}{l_i r}\left[\frac{n_i}{g_i}E(k_i) - K(k_i)\right]$$

$$s_z = \sum_{i=1}^{3}\frac{2\alpha_{i1}\bar{D}_i s_i}{l_i}\left[\frac{o_i}{g_i}E(k_i) - K(k_i)\right]$$

$$E_r = \sum_{i=1}^{3}\frac{2\alpha_{i2}\bar{D}_i z_i}{l_i r}\left[\frac{n_i}{g_i}E(k_i) - K(k_i)\right]$$

$$E_z = -\sum_{i=1}^{3}\frac{2\alpha_{i2}\bar{D}_i s_i}{l_i}\left[\frac{o_i}{g_i}E(k_i) - K(k_i)\right]$$

$$\sigma_r = (c_{11}-c_{12})\sum_{i=1}^{3}\frac{2\bar{D}_i}{l_i r^2}\left[\frac{p_i}{g_i}E(k_i) - (r_0^2+z_i^2)K(k_i)\right] + \sum_{i=1}^{3}\frac{2\xi_i\bar{D}_i}{l_i}\left[\frac{p_i}{g_i}E(k_i) - K(k_i)\right]$$

$$\sigma_\theta = -(c_{11}-c_{12})\sum_{i=1}^{3}\frac{2\bar{D}_i}{l_i r^2}\left[\frac{p_i+o_i r^2}{g_i}E(k_i) - n_i K(k_i)\right] + \sum_{i=1}^{3}\frac{2\xi_i\bar{D}_i}{l_i}\left[\frac{o_i}{g_i}E(k_i) - K(k_i)\right]$$

$$\sigma_m = \sum_{i=1}^{3} \frac{2\vartheta_{im}\bar{D}_i}{l_i}\left[\frac{o_i}{g_i}E(k_i)-K(k_i)\right]$$

$$\tau_{rm} = -\sum_{i=1}^{3} \frac{2\omega_{im}\bar{D}_i z_i}{l_i r}\left[\frac{n_i}{g_i}E(k_i)-K(k_i)\right] \tag{12}$$

(3) Solution for uniform ring loading in θ direction with line density of S_l.

Consider an infinitesimal arc element $r_0\,\mathrm{d}\theta$ at point A. The forces acting on the element are $-S_l r_0 \sin\theta\,\mathrm{d}\theta$ in x direction and $S_l r_0 \cos\theta\,\mathrm{d}\theta$ in y direction. By means of the superposition principle the displacement functions at point B can be obtained from eqns (7) and (8).

$$\mathrm{d}\psi_0 = -\frac{\tilde{D}_0(rr_0\cos\theta-r_0^2)}{\tilde{T}_0}\mathrm{d}\theta, \quad \mathrm{d}\psi_i = -\frac{\tilde{D}_i r r_0 \sin\theta}{\tilde{T}_i}\mathrm{d}\theta \tag{13}$$

where $\tilde{D}_i = S_l D_i/T$ $(i = 0,1,2,3)$.

Integrating the expressions above with respect to θ in the interval of $0 \sim 2\pi$ gives the displacement functions as follows:

$$\psi_0 = 2\tilde{D}_0\left\{\pi[1-\mathrm{sign}(r-r_0)]s_i|z| + l_0E(k_0) - \frac{r^2-r_0^2}{l_0}K(k_0) - \frac{(r-r_0)z_0^2}{(r+r_0)l_0}\Pi(d,k_0)\right\}$$

$$\psi_i = 0, \quad (i = 1,2,3) \tag{14}$$

From eqn (D1), we get the expressions of displacements and electric potential.

$$u_r = 0, \quad u_\theta = \frac{2\tilde{D}_0}{l_0 r}[l_0^2E(k_0)-n_0K(k_0)], \quad w_m = 0. \tag{15}$$

In Ding et al. (1997), B_i in eqn (B3) and C_i in eqn (B7) have the following forms: $B_i = PB_i^P + QB_i^Q$ and $C_i = PC_i^P + QC_i^Q$. Correspondingly, in what follows we have $\bar{B}_i = P_lB_i^P + Q_lB_i^Q$ and $\bar{C}_i = P_lC_i^P + Q_lC_i^Q$, as well as $\bar{E}_i = T_lE_i/T$, $\tilde{E}_i = S_lE_i/T$, $\bar{G}_i = T_lG_i/T$ and $\tilde{G}_i = S_lG_i/T$, etc.

2.2. $s_1 \neq s_2 = s_3$

By use of the displacement functions given in Appendix B of Ding et al. (1997) and eqn (A6), the general solutions for displacements in the case of multiple roots given in Appendix A, applying the same procedure as explained in Section 2.1 leads to the fundamental solutions of axisymmetric problems.

(1) Solution for uniform ring loading in z direction with line density of P_l and uniform charge with line density of Q_l.

$$u_r = -\sum_{i=1}^{2} \frac{2\bar{B}_i r_0 z_i}{l_i r}\left[K(k_i)+\frac{r-r_0}{r+r_0}\Pi(d,k_i)\right]+\frac{2\bar{B}_3 r_0 z_i}{l_2 r}\left[\frac{f_2}{g_2}E(k_2)-K(k_2)\right]$$

$$u_\theta = 0$$

$$w_m = \sum_{i=1}^{2} \frac{4\alpha_{im}\bar{B}_i r_0}{l_i}K(k_i)-\frac{4\alpha_{2m}\bar{B}_3 r_0 z_2^2}{l_2 M_2}E(k_2)+\frac{4\alpha_{4m}\bar{B}_3 t_0}{l_2}K(k_2) \tag{16}$$

(2) Solution for uniform ring loading in r direction with line density of T_l.

$$u_r = -\sum_{i=1}^{2} \frac{2\bar{E}_i}{l_i r}[l_i^2 E(k_i)-n_i K(k_i)]+\frac{2\bar{E}_3 z_2^2}{l_2 r}\left[\frac{n_2}{g_2}E(k_2)-K(k_2)\right]$$

$$u_\theta = 0$$

$$w_m = -\sum_{i=1}^{2} \frac{2\alpha_{im}\bar{E}_i z_i}{l_i}\left[K(k_i)-\frac{r-r_0}{r+r_0}\Pi(d,k_i)\right]-\frac{2\alpha_{2m}\bar{E}_3 z_2}{l_2}\left[\frac{o_2}{g_2}E(k_2)-K(k_2)\right]$$
$$+\frac{2\alpha_{4m}\bar{E}_3 z_2}{l_2}\left[K(k_2)-\frac{r-r_0}{r+r_0}\Pi(d,k_2)\right] \tag{17}$$

(3) Solution for ring loading in θ direction line density of S_l.

The expressions of displacement are the same as those in the case of different roots except for replacing $\tilde{D}_0$ with $\tilde{E}_0$.

2.3. $s_1 = s_2 = s_3$

(1) solution for ring loading in z direction with line density of P_l and uniform charge with line density of Q_l.

$$u_r = -\frac{2\bar{C}_1 r_0 z_1}{l_1 r}\left[K(k_1)+\frac{r-r_0}{r+r_0}\Pi(d,k_1)\right]+\frac{2\bar{C}_2 r_0 z_1}{l_1 r}\left[\frac{f_1}{g_1}E(k_1)-K(k_1)\right]$$
$$-\frac{2\bar{C}_3 r_0 z_1^3}{l_1^3 g_1 r}\left[\frac{p_1-7o_1 r^2}{g_1}E(k_1)-f_1 K(k_1)\right]$$

$$u_\theta = 0$$

$$w_m = \frac{4(\alpha_{1m}\bar{C}_1+\alpha_{4m}\bar{C}_2+\alpha_{5m}\bar{C}_3)r_0}{l_1}K(k_1)-\frac{4(\alpha_{1m}\bar{C}_2+2\alpha_{4m}\bar{C}_3)r_0 z_1^2}{l_1 g_1}E(k_1)$$
$$-\frac{4\alpha_{2m}\bar{C}_3 r_0 z_1^2}{l_1^3 g_1}\left[\frac{f_1^2-4z_1^2(r_0^2+z_1^2)}{g_1}E(k_1)+z_1^2 K(k_1)\right] \tag{18}$$

(2) Solution for ring loading in r direction with line density of T_l.

$$u_r = -\frac{2\bar{G}_1}{l_1 r}[l_1^2 E(k_1) - n_1 K(k_1)] + \frac{2\bar{G}_2 z_1^2}{l_1 r}\left[\frac{n_1}{g_1}E(k_1) - K(k_1)\right]$$

$$+\frac{2\bar{G}_3 z_1^2}{l_1^3 g_1^2 r}\{[l_1^2 g_1(n_1+z_1^2) - 2z_1^2(n_1^2+4r^2r_0^2)]E(k_1) - (l_1^2 g_1 - n_1 z_1^2)g_1 K(k_1)]\}$$

$$u_\theta = 0$$

$$w_m = -\frac{2(\alpha_{1m}\bar{G}_1 - \alpha_{4m}\bar{G}_2 - \alpha_{5m}\bar{G}_3)z_1}{l_1}\left[K(k_1) - \frac{r-r_0}{r+r_0}\Pi(d,k_1)\right]$$

$$-\frac{2(\alpha_{1m}\bar{G}_2 + 2\alpha_{4m}\bar{G}_3)z_1}{l_1}\left[\frac{o_1}{l_1}E(k_1) - K(k_1)\right]$$

$$-\frac{2\alpha_{2m}\bar{G}_3 z_1^3}{l_2^3 g_2^2}[(l_1^2 g_1 - 2n_1^2 + q_1)E(k_1) + o_1 g_1 K(k_1)] \tag{19}$$

(3) Solution for ring loading in θ direction with line density of S_l.

The expressions of displacements are the same as those in the case of mutually different roots except replacing $\tilde{D}_0$ with $\tilde{G}_0$.

Finally, it should be noted that when the displacement functions for axisymmetric problems are obtained first, the displacements and electric potential can be derived from displacement functions by using eqns (D2) and (D3) of Appendix D as in the case of eqns (16–19).

3. Solutions for uniform ring loading in a bimaterial infinite body

Ding et al. (1997) also gave Green's functions for a bimaterial infinite body in three possible cases of characteristic roots s_i. Assume that uniform ring loading acts on the plane $z = h$ in the bimaterial infinite body. Applying the same procedure as that for homogeneous infinite body, again, the solutions for uniform ring loading in a bimaterial infinite body can be obtained by integration. In what follows, the solution for the case of $s_1 \neq s_2 \neq s_3 \neq s_1$ will be presented.

(1) Solution for ring loading in the z direction with line density of P_l and uniform charge with line density of Q_l.

In the region $z \geqslant 0$, we have

$$u_r = -\sum_{i=1}^{3}\frac{2\bar{A}_i r_0 \bar{z}_{ii}}{\bar{l}_{ii} r}\left[K(\bar{k}_{ii}) + \frac{r-r_0}{r+r_0}\Pi(d,\bar{k}_{ii})\right] - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{2\bar{A}_{ij} r_0 z_{ij}}{l_{ij} r}\left[K(k_{ij}) + \frac{r-r_0}{r+r_0}\Pi(d,k_{ij})\right]$$

$$u_\theta = 0$$

$$w_m = \sum_{i=1}^{3}\frac{4\alpha_{im}\bar{A}_i r_0}{\bar{l}_{ii}}K(\bar{k}_{ii}) + \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{4\alpha_{im}\bar{A}_{ij} r_0}{l_{ij}}K(k_{ij}) \tag{20a}$$

In the region $z \leqslant 0$, we have:

$$u_r = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{2\bar{A}'_{ij}r_0 z'_{ij}}{l'_{ij}r}\left[K(k'_{ij})+\frac{r-r_0}{r+r_0}\Pi(d,k'_{ij})\right],\quad u_\theta = 0$$

$$w_m = -\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{4\alpha'_{im}\bar{A}'_{ij}r_0}{l'_{ij}}K(k'_{ij}) \tag{20b}$$

(2) Solution for ring loading in r direction with line density of T_l.
In the region $z \geqslant 0$, we have:

$$u_r = -\sum_{i=1}^{3}\frac{2\bar{D}_i}{\bar{l}_{ii}r}[\bar{l}_{ii}^2E(\bar{k}_{ii})-\bar{n}_{ii}K(\bar{k}_{ii})]-\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{2\bar{D}_{ij}}{l_{ij}r}[l_{ij}^2E(k_{ij})-n_{ij}K(k_{ij})]$$

$$u_\theta = 0$$

$$w_m = \sum_{i=1}^{3}\frac{2\alpha_{im}\bar{D}_i\bar{z}_{ii}}{\bar{l}_{ii}}\left[K(\bar{k}_{ii})-\frac{r-r_0}{r+r_0}\Pi(d,\bar{k}_{ii})\right]$$

$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{2\alpha_{im}\bar{D}_{ij}z_{ij}}{l_{ij}}\left[K(k_{ij})-\frac{r-r_0}{r+r_0}\Pi(d,k_{ij})\right] \tag{21a}$$

In the region $z \leqslant 0$, we have:

$$u_r = -\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{2\bar{L}'_{ij}}{l'_{ij}r}[l'^2_{ij}E(k'_{ij})-n'_{ij}K(k'_{ij})],\quad u_\theta = 0$$

$$w_m = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{2\alpha'_{im}\bar{L}'_{ij}z'_{ij}}{l'_{ij}}\left[K(k'_{ij})-\frac{r-r_0}{r+r_0}\Pi(d,k'_{ij})\right] \tag{21b}$$

(3) Solution for ring loading in θ direction with line density of S_l.
In the region $z \geqslant 0$, we have:

$$u_r = 0,\quad w_m = 0$$

$$u_\theta = \frac{2\tilde{D}_0}{\bar{l}_{00}r}[\bar{l}_{00}^2E(\bar{k}_{00})-\bar{n}_{00}K(\bar{k}_{00})]+\frac{2\tilde{D}_{00}}{l_{00}r}[l_{00}^2E(k_{00})-n_{00}K(k_{00})] \tag{22a}$$

In the region $z \leqslant 0$, we have:

$$u_r = 0,\quad w_m = 0$$

$$u_\theta = \frac{2\tilde{L}'_{00}}{l'_{00}r}[l'^2_{00}E(k'_{00})-n'_{00}K(k'_{00})] \tag{22b}$$

In the derivation of eqns (20)–(22), we have made use of the displacement and electric potential expressions eqns (28) and (30) as well as expressions of displacement function eqns (35), (51) and (53) in Ding et al. (1997). In addition, eqns (31)–(34) are used to solve for A_{ij} and A'_{ij}, and eqns (55)–(60) for D_{ij} and L'_{ij}. Obviously, these constants can be rewritten to the following forms:

$A_{ij} = PA^P_{ij} + QA^Q_{ij}$, $A'_{ij} = PA'^P_{ij} + QA'^Q_{ij}$, $D_{ij} = TD^T_{ij}$ and $L'_{ij} = TL'^T_{ij}$. Therefore, those constants appearing in eqns (20)–(22) are all of the following forms: $\bar{A}_{ij} = P_l A^P_{ij} + Q_l A^Q_{ij}$, $\bar{A}'_{ij} = P_l A'^P_{ij} + Q_l A'^Q_{ij}$, $\bar{D}_{ij} = T_l D^T_{ij}$, $\bar{L}'_{ij} = T_l L'^T_{ij}$, $\tilde{D}_{ij} = S_l D^T_{ij}$ and $\tilde{L}'_{ij} = S_l L'^T_{ij}$. For those solutions in which displacement functions are obtained first, displacements and electric potential can be calculated by eqn (D1) of Appendix D correspondingly.

4. Solutions for uniform ring loading in semi-infinite body

Assume that uniform ring loading acts on the plane $z = h$ in a semi-infinite body. The solutions for displacements and electric potential to the extended Mindlin problem in the case of $s_1 \neq s_2 \neq s_3 \neq s_1$ have the forms of eqns (20a), (21a) and (22a). Yet, coefficients A^P_{ij}, A^Q_{ij} and D^T_{ij} in $\bar{A}_{ij}$, $\bar{D}_{ij}$ and $\tilde{D}_{ij}$ should be determined by eqns (61) and (62) of Ding et al. (1997).

5. Numerical examples

Taking PZT-4 as an example, some quantities of eqns (5), (6), (11) and (12) are calculated, and comparison of the calculated results with those of FEM is made. Material constants and characteristic roots of piezoelectric material PZT-4 are listed in Table 1. Take a cylinder with square meridional plane of side length 1000 m as the region to be studied as shown in Fig. 2. Halves of unit axial force (1 N/m), unit charge density (1 C/m) and unit radial force (1 N/m) are applied at the ring that is 10 m away from the z axis. The plane of OE is the plane of symmetry. The axial displacement, radial displacement and electric potential are set to be zero at the outer boundary DE and CD when FEM calculation is performed. Region 1 and region 2 are divided into 400 elements and 500 elements, respectively, which amount to 2801 nodes and 8403 degree-of-freedoms. Isoparametric elements with eight nodes are adopted. Comparison of calculated results of certain quantities on the plane $z = 0$ by FEM and those of fundamental solutions is listed in Figs 3–15, where FSL stands for results of fundamental solutions and FEM for those of Finite Element Method.

Table 1
Material constants of piezoelectric material PZT-4

Elastic constants (N/m^2)	Piezoelectric constants (C/m^2)	Dielectric constants (C/Vm)	Characteristic roots
$c_{11} = 13.9 \times 10^{10}$	$e_{31} = 5.2$	$\varepsilon_{11} = 6.46 \times 10^{-9}$	$s_1 = 1.203962$
$c_{12} = 7.78 \times 10^{10}$	$e_{33} = 15.1$	$\varepsilon_{33} = 5.62 \times 10^{-9}$	$s_2 = 1.069818$
$c_{13} = 7.43 \times 10^{10}$	$e_{15} = 12.7$		$+1.997586$ I
$c_{33} = 11.5 \times 10^{10}$			$s_3 = 1.046767$
$c_{44} = 2.56 \times 10^{10}$			-1.997586 I

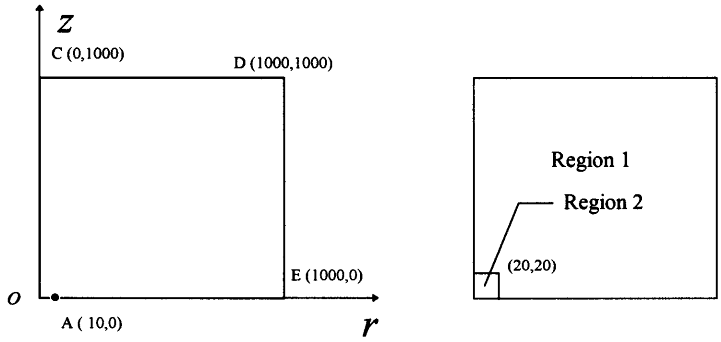

Fig. 2. FEM model.

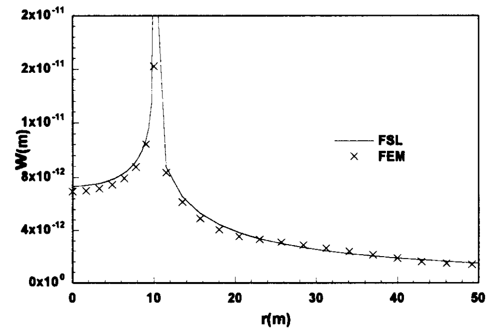

Fig. 3. w caused by axial force.

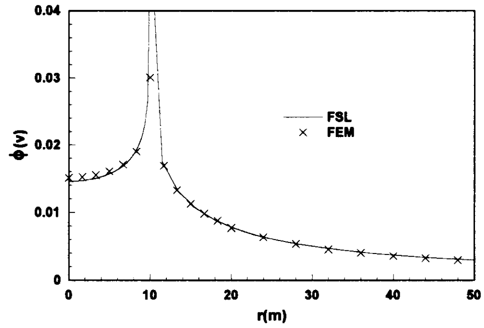

Fig. 4. ϕ caused by axial force.

6. Conclusions

(1) The axisymmetric fundamental solutions for transversely isotropic piezoelectric materials are derived and solutions for all three possible cases of characteristic roots s_i are explicitly given. Comparison of the calculated results of the fundamental solutions with those of FEM shows

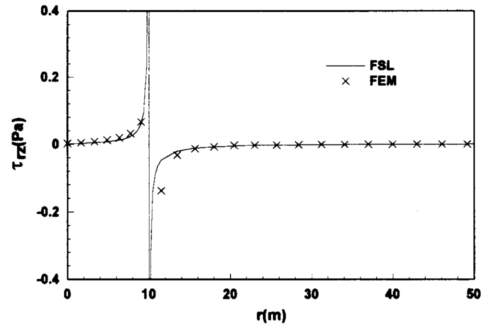

Fig. 5. τ_{rz} caused by axial force.

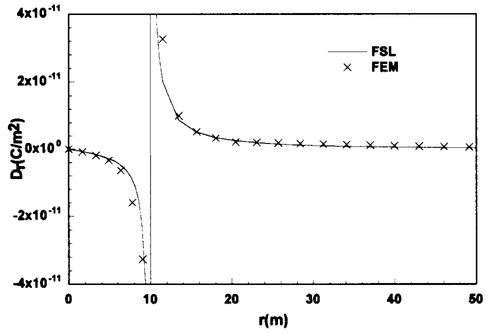

Fig. 6. D_r caused by axial force.

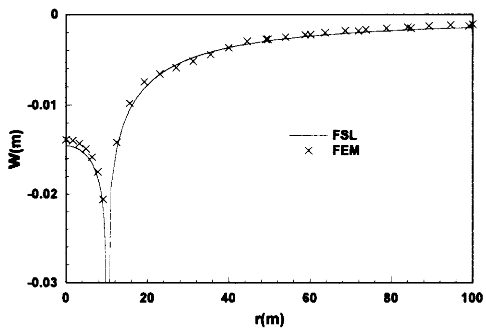

Fig. 7. w caused by electric charge.

good agreement between them. In eqn (5), that is, the solution for ring loading in z direction with line density of P_l and uniform charge of line density Q_l in the case of $s_1 \neq s_2 \neq s_3 \neq s_1$, assume that $2\pi r_0 P_l = P$, $2\pi r_0 Q_l = Q$, and let r_0 approach zero while keeping P and Q constant, then eqn (5) will be reduced to eqns (12)–(14) of Ding et al. (1997) that are expressed in cylindrical coordinates.

(2) For axisymmetric problems of transversely isotropic piezoelectric semi-infinite body and bima-

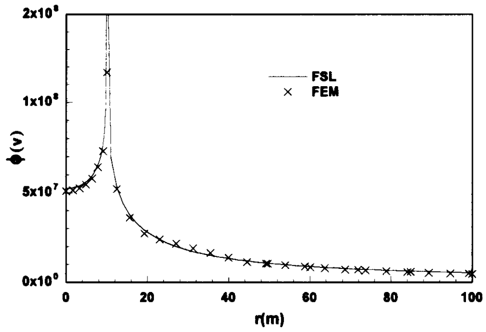

Fig. 8. ϕ caused by electric charge.

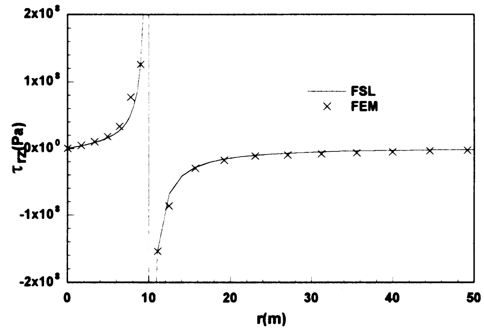

Fig. 9. τ_{rz} caused by electric charge.

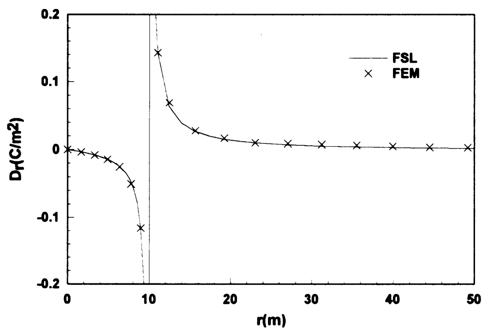

Fig. 10. D_r caused by electric charge.

terial infinite body, Green's functions in the case of $s_1 \neq s_2 \neq s_3 \neq s_1$ are present. Green's functions corresponding to the cases of multiple characteristic roots s_i can also be obtained by integration from the related equations of Ding et al. (1997).

(3) Compared with the fundamental solutions for isotropy, the fundamental solutions given in this paper involve not only elliptic integrals of first and second kind, but also elliptic integral

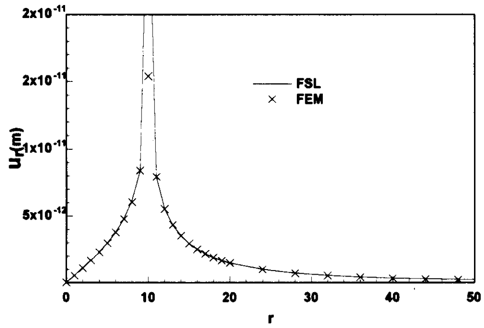

Fig. 11. u_r caused by radial force.

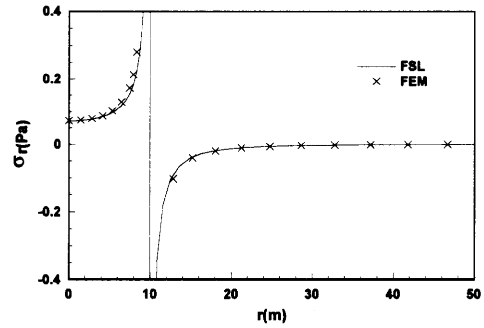

Fig. 12. σ_r caused by radial force.

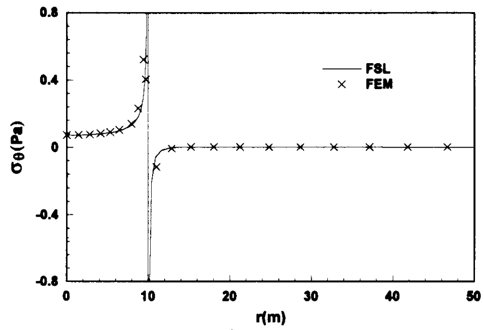

Fig. 13. σ_θ caused by radial force.

of third kind, which also occurs in the problem of transversely isotropic elastic body (Hanson and Yang Wang, 1997).

(4) Making full use of the results for the case of $s_1 \neq s_2 \neq s_3 \neq s_1$ can facilitate the solution of Green's functions in the cases of multiple roots because in the displacement functions for the cases of multiple roots, terms different from those in the case of $s_1 \neq s_2 \neq s_3 \neq s_1$ are, in

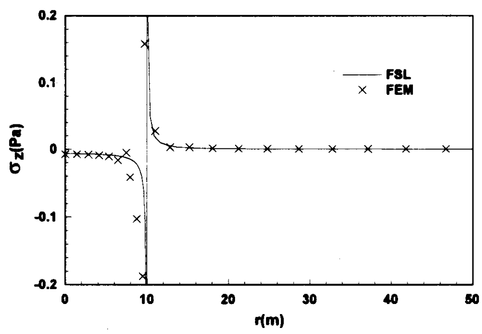

Fig. 14. σ_z caused by radial force.

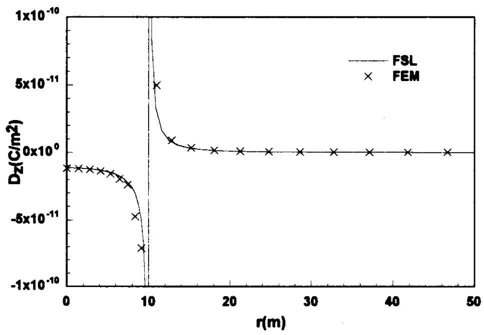

Fig. 15. D_z caused by radial force.

fact, in direct proportion to partial derivatives of their corresponding original displacement functions with respect to z. Therefore, the desired results can be readily obtained by proper use of eqns (B4)–(B7) and eqns (D1)–(D3) in appendices, and complicated integration could be obviated.

Appendix A: Notations

For convenience of reading, frequently used notations in the paper are listed as follows:

$$z_i = s_i z \quad l_i^2 = (r+r_0)^2 + z_j^2$$

$$f_i = -r^2 + r_0^2 + z_i^2 \quad g_i = (r-r_0)^2 + z_i^2$$

$$k_i = 2\sqrt{rr_0}/l_i \quad d = -4rr_0/(r+r_0)^2$$

$$n_i = r^2 + r_0^2 + z_i^2 \quad o_i = r^2 - r_0^2 + z_i^2$$

$$p_i = r^2(z_i^2 - r_0^2) + (r_0^2 + z_i^2)^2 \quad q_i = 8r_0^2(r_0^2 + z_i^2)$$

$$b^2 = 4rr_0 \quad c^2 = 2(r^2+r_0^2)$$

$$h_i = s_i h \quad z'_i = s'_i z_i$$

$$\bar{z}_{ij} = z_i - h_j \quad z_{ij} = z_i + h_j$$

$$z'_{ij} = z'_i - h_j \quad \bar{l}^2_{ij} = (r+r_0)^2 + \bar{z}^2_{ij}$$

$$l^2_{ij} = (r+r_0)^2 + z^2_{ij} \quad l'^2_{ij} = (r+r_0)^2 + z'^2_{ij}$$

$$\bar{k}_{ij} = 2\sqrt{rr_0}/\bar{l}_{ij} \quad k_{ij} = 2\sqrt{rr_0}/l_{ij}$$

$$k'_{ij} = 2\sqrt{rr_0}/l'_y \quad \bar{n}_{ij} = r^2 + r_0^2 + \bar{z}^2_{ij}$$

$$n_{ij} = r^2 + r_0^2 + z^2_{ij} \quad n'_{ij} = r^2 + r_0^2 + z'^2_{ij}$$

Appendix B: Elliptical integrals and their differentiation formulas

(1) Three kinds of complete Legendre elliptic integrals

Complete elliptic integral of the first kind:

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} \tag{B1}$$

Complete elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\psi}\, d\psi \tag{B2}$$

Complete elliptic integral of the third kind:

$$\Pi(\rho,k) = \int_0^{\pi/2} \frac{d\psi}{(1+\rho\sin^2\psi)\sqrt{1-k^2\sin^2\psi}} \tag{B3}$$

(2) Differentiation formulas

$$\frac{dK(k)}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k} \tag{B4}$$

$$\frac{dE(k)}{dk} = \frac{E(k)}{k} - \frac{K(k)}{k} \tag{B5}$$

$$\frac{\partial\Pi(\rho,k)}{\partial k} = \frac{k}{(k^2+\rho)}\left[\frac{E(k)}{1-k^2} - \Pi(\rho,k)\right] \tag{B6}$$

$$\frac{\partial \Pi(\rho,k)}{\partial \rho} = \frac{1}{2\rho(1+\rho)(k^2+\rho)}[\rho E(k)-(\rho+k^2)K(k)+(k^2-\rho^2)\Pi(\rho,k)] \tag{B7}$$

Appendix C: Some definite integrals appearing in the fundamental solutions

First

$$\int_0^{2\pi} \frac{1}{c^2-b^2\cos\theta}\mathrm{d}\theta = \frac{2\pi}{\sqrt{c^4-b^4}} = \frac{\pi}{|r^2-r_0^2|} \tag{C1}$$

$$\int_0^{2\pi} \frac{1}{\tilde{R}_i}\mathrm{d}\theta = \frac{4}{l_i}K(k_i) \tag{C2}$$

$$\int_0^{2\pi} \frac{1}{(c^2-b^2\cos\theta)\tilde{R}_i}\mathrm{d}\theta = \frac{4}{l_i(b^2+c^2)}\Pi(d,k) = \frac{2}{l_i(r+r_0)^2}\Pi(d,k_i) \tag{C3}$$

$$\int_0^{2\pi} \tilde{R}_i\,\mathrm{d}\theta = 4l_iE(k_i) \tag{C4}$$

Then, note that

$$\frac{\cos\theta}{\tilde{R}_i} = \frac{2n_i}{b^2}\frac{1}{\tilde{R}_i} - \frac{2}{b^2}\tilde{R}_i \tag{C5}$$

$$\begin{aligned}\frac{1}{\tilde{T}_i} &= \frac{1}{\sqrt{n_i-2rr_0\cos\theta+s_i|z|}}\\ &= -\frac{2s_i|z|}{c^2-b^2\cos\theta}+\frac{1}{\tilde{R}}+\frac{2z_i^2}{(c^2-b^2\cos\theta)\tilde{R}_i}\end{aligned} \tag{C6}$$

Accordingly, definite integrals $\int_0^{2\pi}(\cos\theta\,\mathrm{d}\theta/\tilde{R}_i)$ and $\int_0^{2\pi}(\mathrm{d}\theta/\tilde{T}_i)$ can be calculated by using eqns (C1)–(C4).

Finally, note that

$$\frac{\cos\theta}{\tilde{T}_i} = \frac{2s_i|z|}{b^2} - \frac{2}{b^2}\tilde{R}_i + \frac{c^2}{b^2}\frac{1}{\tilde{T}_i} \tag{C7}$$

$$\frac{r-r_0\cos\theta}{\tilde{R}_i\tilde{T}_i} = \frac{r}{s_i|z|}\left(\frac{1}{\tilde{R}_i}-\frac{1}{\tilde{T}_i}\right) - \frac{r_0}{s_i|z|}\left(\frac{\cos\theta}{\tilde{R}_i}-\frac{\cos\theta}{\tilde{T}_i}\right) \tag{C8}$$

Equation (C8) can be decomposed using eqn (C7), which, in turn, can be decomposed by use of eqns (C5) and (C6). Therefore, integrals $\int_0^{2\pi}(\cos\theta\,\mathrm{d}\theta/\tilde{T}_i)$ and $\int_0^{2\pi}[(r-r_0\cos\theta)\,\mathrm{d}\theta/\tilde{R}_i\tilde{T}_i]$ can be expressed by combinations of elliptic functions.

Appendix D: General solution of axisymmetric problems in cylindrical coordinates

For convenience of reference, we transform three sets of general solution of three-dimensional problem eqns (2), (A6) and (A8) of Ding et al. (1997) into cylindrical coordinates and present the following three sets of general solution of axisymmetric problems.

(1) $s_1 \neq s_2 \neq s_3 \neq s_1$

$$u_r = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial r}, \quad u_\theta = \frac{\partial \psi_0}{\partial r}, \quad w_m = \sum_{i=1}^{3} \alpha_{im} \frac{\partial \psi_i}{\partial z_i}, \quad (m = 1, 2) \tag{D1}$$

(2) $s_1 \neq s_2 = s_3$

$$u_r = \sum_{i=1}^{2} \frac{\partial \psi_i}{\partial r} + z_2 \frac{\partial \psi_3}{\partial r}, \quad u_\theta = \frac{\partial \psi_0}{\partial r}$$

$$w_m = \sum_{i=1}^{2} \alpha_{im} \frac{\partial \psi_i}{\partial z_i} + \alpha_{2m} z_2 \frac{\partial \psi_3}{\partial z_2} + \alpha_{4m} \psi_3, \quad (m = 1, 2) \tag{D2}$$

(3) $s_1 = s_2 = s_3$

$$u_r = \frac{\partial \psi_1}{\partial r} + z_1 \frac{\partial \psi_2}{\partial r} + z_1^2 \frac{\partial \psi_3}{\partial r \, \partial z_1}, \quad u_\theta = \frac{\partial \psi_0}{\partial r}$$

$$w_m = \alpha_{1m} \frac{\partial \psi_1}{\partial z_1} + \alpha_{1m} z_1 \frac{\partial \psi_2}{\partial z_1} + \alpha_{4m} \psi_2 + \alpha_{2m} z_1^2 \frac{\partial^2 \psi_3}{\partial z_1^2} + 2\alpha_{4m} z_1 \frac{\partial \psi_3}{\partial z_1} + \alpha_{5m} \psi_3, \quad (m = 1, 2) \tag{D3}$$

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